Learning Weighted Automata over Principal Ideal Domains

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$\mathtt{L}^\star$ setup for DFAs

Finite alphabet A

System behaviour captured by a **regular language** $\mathcal{L} \subseteq A^*$

$\mathtt{L}^\star$ learns *minimal* DFA for $\mathcal{L}$

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L* setup for DFAs

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- **Membership queries**

$$\boxed{w \in \mathcal{L}?}$$

- **Equivalence queries**

$$\boxed{\mathcal{L}(H) = \mathcal{L}?}$$

Negative result $\implies$ *counterexample*

$\mathrm{L}^\star$ algorithm (variation) for DFAs

$S, E \subseteq A^*$ induce a table

$$
\begin{array}{r|cc}
 & \overbrace{\varepsilon \quad \mathrm{a}}^{E} \\
\hline
\varepsilon & 1 \quad 0 \\
\mathrm{a} & 0 \quad 1 \\
\mathrm{aa} & 1 \quad 0 \\
\hline
\mathrm{aaa} & 0 \quad 1
\end{array}
$$

		ε	a
S	ε	1	0
	a	0	1
	aa	1	0
$S \cdot A$	aaa	0	1

$\mathcal{L} = \{a^n \mid n \text{ is even}\}$

$aa \cdot a \notin \mathcal{L}$

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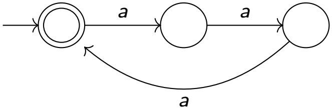
$\mathtt{L}^\star$ for DFAs, example

$$a^n \in \mathcal{L} \iff n \equiv 0 \pmod 3$$



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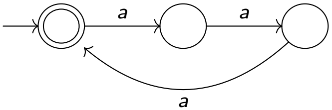
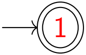


	ε
ε	1
a	0



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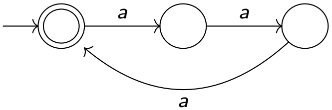

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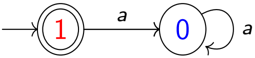

Counterexample: aaa

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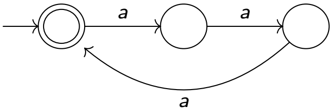
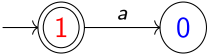


	ε	a	aa	aaa
ε	1	0	0	1
a	0	0	1	0
aa	0	1	0	0



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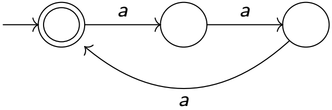

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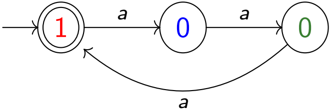

DFAs vs WFAs

$\mathcal{S}$ semiring, FQ free semimodule over Q

$\boxed{\text{DFA}}$	$\boxed{\text{WFA}}$
initial state in Q	initial state in FQ
$Q \to 2 \times Q^A$	$Q \to \mathcal{S} \times (FQ)^A$

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$$\begin{array}{c} Q \\ \downarrow \\ 2 \times Q^A \end{array} \qquad \begin{array}{c} Q \\ \downarrow \\ \mathcal{S} \times (FQ)^A \end{array}$$

Interpretation: **weighted language** $A^* \to \mathcal{S}$

- multiply weights along paths and with final output
- sum over paths

Algorithm setup for WFAs

Membership queries:

return output value associated with word

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output values in $\mathcal{S}$ instead of $0, 1$

Closedness:

each lower row a linear combination of upper rows

General (weighted) L*

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Repeat until no more counterexamples:

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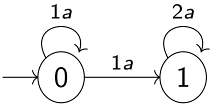
Example over $\mathbb{Q}$

$$\mathcal{L}(a^n) = 2^n - 1$$



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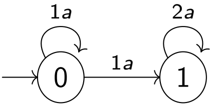
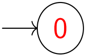


	ε
ε	0
a	1



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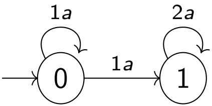
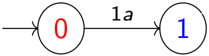


	ε
ε	0
a	1
aa	3



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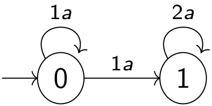

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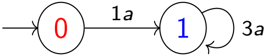

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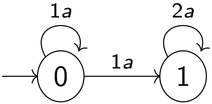
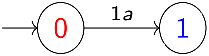


	ε	a	aa	aaa
ε	0	1	3	7
a	1	3	7	15
aa	3	7	15	31



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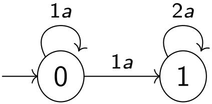

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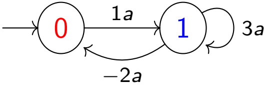

Termination of the general algorithm

Algorithm terminates assuming

- **progress measure with bound**

  Number, increases when rows separate via extra column

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- **ascending chain condition** on Hankel matrix (table (A^*, A^*))

  Subsemimodule chains converge: if

  $$S_1 \subseteq S_2 \subseteq \cdots \subseteq H$$

  are subsemimodules, then there exists $n \in \mathbb{N}$ s.t.

  $$S_n = S_{n+1} = S_{n+2} = \cdots$$

Termination argument

Assume

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Bounded progress measure $\implies$ finitely many counterexamples

Main ingredients for effective terminating algorithm

1. **Progress measure with bound**
2. **Ascending chain condition** on Hankel matrix
3. **Procedure to determine/fix closedness:**
   solvability of finite system of linear equations

WFAs over field: no problem

1. Progress measure and bound
   - Dimension of vector space spanned by table
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3. Procedure to determine/fix closedness
   - Gaussian elimination

WFAs over finite semiring: naive algorithm

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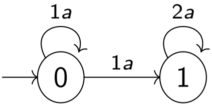
   - Try all linear combinations of rows

WFAs over $\mathbb{N}$: termination issue

$$\mathcal{L}(a^n) = 2^n - 1$$



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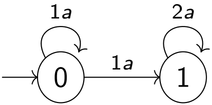

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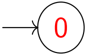
ε	0	1
a	1	3



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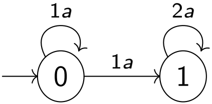

	ε	a
ε	0	1
a	1	3
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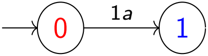

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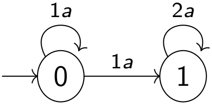

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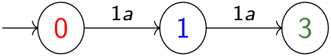

Analysis: overweight finite automaton

Infinite chain of strict semimodule embeddings

- Ascending chain condition fails
- Hankel matrix not even finitely generated

Contribution: WFAs over PID

Principal ideal domain $=$ integral domain with all ideals principal

Integral domain: commutative ring,

$$ab = 0 \implies a = 0 \lor b = 0$$

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Examples: $\mathbb{Z}$, $\mathbb{Z}[i]$, $K[x]$ for K a field

PID free module properties

A module is free if and only if it is **torsion free**:

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If a finitely generated free module is a quotient of another, its rank is smaller or equal

Progress measure for PIDs

Table modules are torsion free and thus free

Measure: **rank of table module**

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Bound: **Hankel matrix rank**

Progress measure for PIDs

Table modules are torsion free and thus free

Measure: **rank of table module**

Bound: **Hankel matrix rank**

Progress (general fact): for X, Y finite sets and

- $FX \xrightarrow{f} FY$ a surjective homomorphism
- that identifies some elements

we have $|X| > |Y|$

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     (basis of the union of the chain included in a chain element)

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2. Ascending chain condition
   - Yes :)
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3. Procedure to determine/fix closedness
   - Solve equations via Smith normal form (exists for PIDs)

Future work

Even bigger classes of semirings

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Stronger negative results

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Even bigger classes of semirings

Stronger negative results

Conditions on the monad